NOTATION

c, dimensionless (in inlet concentration units) concentration of sorbed gas or liquid in mobile phase; a, dimensionless (in inlet concentration units) concentration of matter in sorbent per unit volume of the mobile phase; f(c), isotherm equation of sorption; x, coordinate; η, dimensionless coordinate; t, time; τ, τ', dimensionless times; u, mean flow velocity; D, lengthwise diffusion coefficient; l, kinetic parameter—retardation path; γ, Henry coefficient; ε, isotherm nonlinearity parameter; m, exponent; $x_0(t)$, mobile boundary of concentration c = 1; δ(τ), dimensionless mobile boundary of concentration c = 1; K, constant.

LITERATURE CITED

1. V. V. Rachinskii, Introduction to the General Theory of Dynamics of Sorption and Chromatography [in Russian], Nauka, Moscow (1964).
2. L. Lapidus and N. R. Amundsen, J. Phys. Chem., 56, 984 (1952).
3. S. E. Bresler and Ya. S. Uflyand, Zh. Tekh. Fiz., 28, 1443 (1953).
4. M. S. Safonov and V. A. Poteshnov, Teor. Osn. Khim. Tekhnol., 3, 42 (1969).
5. A. I. Kalinichev and P. P. Zolotarev, Zh. Fiz. Khim., 47, 1583 (1973).
6. A. I. Kalinichev, Zh. Fiz. Khim., 50, 1839 (1976).
7. P. P. Zolotarev, Izv. Akad. Nauk SSSR, Ser. Khim., No. 3 (1969).
8. T. Goodman, in: Problems of Heat Exchange [Russian translation], Atomizdat, Moscow (1967), p. 41.
9. A. I. Kalinichev, Zh. Fiz. Khim., 50, 3135 (1976).

AN APPROXIMATE MATHEMATICAL MODEL OF HEAT AND MASS TRANSFER IN TWO-PHASE FLOW

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We propose an approximate method for studying transport processes in one-dimensional two-phase flow which permits the determination of the system output as a function of input and the system parameters. The error of the method is estimated.

A mathematical description of transport processes in steady two-phase flows characterized both by mixing and the presence of arbitrary sources whose strengths depend only on the potentials of the entities being transported is important for the chemical industry. For simplicity we consider only one-dimensional flows. It is known that such a description cannot generally be given in exact closed analytic form even for one-phase systems. There is evidence, however, that for a certain heuristic reinterpretation of the differential equation to be solved and its boundary conditions, an approximate method of describing the system analytically can be constructed [1].

In the present paper we investigate such a method in a general form suitable for a mathematical description of two-phase heat and mass transfer. The proposed method can be used not only for an approximate analytic study of heat and mass transfer in two interacting phases, but also for an approximate study of heat- and mass-transfer processes taking place simultaneously in a single phase. In addition, this method can in principle be generalized to include an arbitrary number of equations. There then arises the problem of comparing the approximate and exact results. In this paper we restrict ourselves to the most important practical case of two equations.

It is known [2] that transport processes in the systems under consideration can be described in the usual approximation by the equations

$$-F\mathscr{K}\frac{d^2u}{dx^{*2}}+V\frac{du}{dx^*}-f^*(u,\ u')=0, \tag{1}$$

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$$-F'\mathscr{K}'\frac{d^2u'}{dx^{*2}}+V'\frac{du'}{dx^*}-f'^*(u,\ u')=0,$$

where u and u' are potentials describing the extensive quantities whose transport is being investigated, F and F' are the separate transfer cross sections, $\mathscr{K}$ and $\mathscr{K}'$ are mixing coefficients, V and V' are the bulk velocities of the separate phases, f* and f'* are source functions which are positive for a source and negative for a sink. The only condition imposed on the source functions so far is that they must be continuous functions of the potentials. Henceforth we consider the coordinate x* in the range $-\infty < x^* < +\infty$, although the system under investigation lies in the range $0 \leqslant x^* \leqslant L$.

By introducing the dimensionless coordinate $x = x^*/L$ and the dimensionless quantities $A = F\mathscr{K}/VL$ and $A' = F'\mathscr{K}'/V'L$ the system of equations (1) can be reduced to the form

$$-A\frac{d^2u}{dx^2}+\frac{du}{dx}-f(u,\ u')=0,$$
$$-A'\frac{d^2u'}{dx^2}+\frac{du'}{dx}-f'(u,\ u')=0. \qquad (2)$$

These equations are defined in the range $0 \leq x \leq 1$; f and f' are the transformed source functions.

From now on we write all relations in a form corresponding to Eqs. (2). Boundary conditions must be specified for Eqs. (2), and since this is not a trivial problem we consider it first.

To formulate natural boundary conditions for Eqs. (2) it is necessary to investigate mathematical processes in the regions preceding the system entrance and following the exit for appropriate matching conditions. If a direct representation of the functions describing the inflow and outflow of the entities being transported is possible, we obtain the required boundary conditions for the system under study from the matching conditions at the system entrance and exit.

We consider first the conditions at the entrance, i.e., in the range $-\infty < x < 0$. We simplify the problem without loss of generality by considering only the once-through problem. In the range indicated we can write

$$-A_1\frac{d^2u_1}{dx^2}+\frac{du_1}{dx}=0,\quad -A_1'\frac{d^2u_1'}{dx^2}+\frac{du_1'}{dx}=0, \qquad (3)$$

where u and u' are the corresponding potentials, and A_1 and A_1' are dimensionless mixing coefficients in the region $-\infty < x < 0$.

At points sufficiently distant from the matching point x = 0 the potentials must have certain specified values. At the matching point the flow of the extensive quantity described by the given potential, and the potential itself, must be continuous, i.e.,

$$\lim_{x\to-\infty} u_1=u_b,\quad u_1(0)=A_1\left(\frac{du_1}{dx}\right)_{x=0}=u(0)-A\left(\frac{du}{dx}\right)_{x=0}.$$
$$\lim u_1'=u_b',\quad u_1'(0)=A_1'\left(\frac{du_1'}{dx}\right)_{x=0}=u'(0)-A'\left(\frac{du'}{dx}\right)_{x=0}, \qquad (4)$$
$$u_1(0)=u(0),\quad u_1'(0)=u'(0).$$

Similarly, in the range $1 < x < +\infty$ we have

$$-A_2\frac{d^2u_2}{dx^2}+\frac{du_2}{dx}=0,\quad -A_2'\frac{d^2u_2'}{dx^2}+\frac{du_2'}{dx}, \qquad (5)$$

where u_2 and u_2' are potentials and A_2, A_2' are dimensionless mixing coefficients.

Using the continuity conditions at the matching point x = 1, and requiring that the potentials be bounded in the range considered, we obtain

$$u(1)-A\left(\frac{du}{dx}\right)_{x=1}=u_2(1)-A_2\left(\frac{du_2}{dx}\right)_{x=1},$$

$$u'(1) - A'\left(\frac{du'}{dx}\right)_{x=1} = u_2'(1) - A_2'\left(\frac{du_2'}{dx}\right)_{x=1},$$
$$u(1) = u_2(1), \quad u'(1) = u_2'(1), \tag{6}$$
$$\lim_{x\to+\infty} u_2 < +\infty, \quad \lim_{x\to+\infty} u_2' < +\infty.$$

We consider now the conclusions which can be drawn with respect to the boundary conditions at the system entrance by using the general solution of Eqs. (3). Their general solution has the form

$$u_1(x) = C_{11}e^{x/A_1} + C_{12}, \quad u_1'(x) = C_{11}'\, e^{x/A_1'} + C_{12}', \tag{7}$$

where the C are arbitrary constants. A comparison of (7) with the first two conditions of (4) shows that $C_{12} = u_b$ and $C_{12}' = u_b'$. The next two conditions of (4) lead to very important formulas which represent the boundary conditions at the entrance to the system under study:

$$u(0) - A\left(\frac{du}{dx}\right)_{x=0} = u_b, \quad u'(0) - A'\left(\frac{du'}{dx}\right)_{x=0} = u_b'. \tag{8}$$

Similarly, the most general solution of (4) has the form

$$u_2(x) = C_{21}e^{x/A_2} + C_{22}, \quad u_2'(x) = C_{21}'\, e^{x/A_2'} + C_{22}', \tag{9}$$

where the C are constants. It follows directly from (6) that $C_{21} = C_{21}' = 0$ and that the required boundary conditions at the system exit are

$$\left(\frac{du}{dx}\right)_{x=1} = \left(\frac{du'}{dx}\right)_{x=1} = 0. \tag{10}$$

We note that the mathematical structure of solutions (7) and (9) is essential in the derivation of the boundary conditions; i.e., full use is made of the fact that the mixing occurring before the system entrance and following its exit, and in the system itself, is described by the diffusion model [3]. This enables us to give a physical interpretation of the boundary conditions derived similar to that generally given in the diffusion model.

Thus, Eqs. (2) must be solved with boundary conditions (8) and (10). As has been pointed out, the solution cannot be found in closed form. To obtain the solutions it would first be necessary to find the operators inverse to the kind of differential operators appearing in Eqs. (2) in which the boundary conditions would also be taken into account. These operators would enable us to obtain solutions by the method of successive approximations.

Since boundary conditions (8) are inhomogeneous, finding the inverse integral operators is not equivalent to finding Green's functions of the differential operators mentioned. The required relations can be determined directly, however, by the method of variation of constants [4]. The basis functions for homogeneous equations formed by the differential operators in (2) are 1 and $e^{x/A}$, and also 1 and $e^{x/A'}$. Accordingly, the solutions of Eqs. (12) can be written in the form

$$u(x) = \Gamma_1(x) + \Gamma_2(x)\, e^{x/A}, \quad u'(x) = \Gamma_1'(x) + \Gamma_2'(x)\, e^{x/A}, \tag{11}$$

where the Γ are functions to be determined.

From the condition that the first derivatives of u and u' in (11) must have the simplest possible form, and from the fact that u and u' must satisfy Eqs. (2), we obtain the following equations for the Γ:

$$\frac{d\Gamma_1}{dx} + \frac{d\Gamma_2}{dx}\, e^{x/A} = 0, \quad \frac{d\Gamma_2}{dx}\, e^{x/A} + f(u, u') = 0,$$
$$\frac{d\Gamma_1'}{dx} + \frac{d\Gamma_2'}{dx}\, e^{x/A'} = 0, \quad \frac{d\Gamma_2}{dx}\, e^{x/A'} + f'(u, u') = 0. \tag{12}$$

By transforming and integrating (12) we obtain the following relations:

$$\Gamma_1(x) = \Gamma_{10} + \int_0^x f(u(\xi), \quad u'(\xi))\, d\xi,$$

$$\Gamma_2(x) = \Gamma_{20} + \int_0^x e^{-\xi/A} f(u(\xi), \quad u'(\xi))\, d\xi,$$

$$\Gamma_1'(x) = \Gamma_{10}' + \int_0^x f'(u(\xi), \quad u'(\xi))\, d\xi, \tag{13}$$

$$\Gamma_2'(x) = \Gamma_{20}' - \int_0^x e^{-\xi/A'} f'(u(\xi), \quad u'(\xi))\, d\xi,$$

where the Γ_0 are constants determined from boundary conditions (8) and (10). Substituting (13) into (11) and the boundary conditions, we obtain

$$\Gamma_{10} = u_b, \quad \Gamma_{20} = \int_0^1 e^{-\xi/A} f(u(\xi), \; u'(\xi))\, d\xi,$$

$$\Gamma_{10}' = u_b', \; \Gamma_{20}' = \int_0^1 e^{-\xi/A'} f'(u(\xi), \; u'(\xi))\, d\xi. \tag{14}$$

Finally, substituting (14) into (13) and (11) and making some transformations, we obtain the following integral equations:

$$u(x) = u_b + \int_0^x f(u(\xi), \quad u'(\xi))\, d\xi + \int_x^1 e^{\frac{x-\xi}{A}} f(u(\xi), \; u'(\xi))\, d\xi,$$

$$u'(x) = u_b' + \int_0^x f'(u(\xi), \; u'(\xi))\, d\xi + \int_x^1 e^{\frac{x-\xi}{A'}} f'(u(\xi), \; u'(\xi))\, d\xi. \tag{15}$$

We note that the integral operators already found act on the source functions on the right-hand sides of the integral equations (15); i.e., we finally obtain a certain transformed form of the initial problem which is expedient to use for several reasons. In the first place, instead of the initial boundary-value problem described by six equations of different types, we obtain two equations of the same type which are mathematically equivalent to the initial problem. As will be seen later, such equations are very useful also for comparing exact and approximate results.

Secondly, Eqs. (15) make possible a successive-approximations solution which is a unique method of solving the problem. Since the integral operators which appear are nonlinear, the relatively well-developed theory of linear operators is not applicable. Equations (15) could be linearized by expanding the source functions f and f' in series but this would change the mathematical meaning of the problem under study. As a result, for example, the estimate of the error in comparing the approximate and linearized solutions would lose its meaning.

Equations (15) enable us in principle to find the exact solution by the iteration method. If trial functions u and u' are substituted into the right-hand sides of (15) and the results obtained on the left-hand sides are again substituted into the right-hand sides, and this is repeated an infinite number of times, the exact solutions u and u' are finally obtained if the iteration process converges.

The heuristic approximate analytic method of solution developed is characterized first by the fact that it is based on the synthesis of several partial results and takes account of most practical requirements. The construction of the approximate method was based on the following considerations:

a) The use of a second-order equation, which is very difficult to study, can be given up. An attempt can be made to compensate the inadequacies of the mathematical description given by a first-order equation by the use of appropriately chosen heuristic parameters or boundary conditions.

b) As the mixing coefficient approaches zero the method of approximations was required to lead to results reflecting all the characteristic features of the exact model without mixing, and as it approaches infinity – to results completely reflecting the characteristic features of the model with complete mixing.

c) In a number of cases it is sufficient to restrict ourselves to finding approximate values of the potentials at the system exit only, i.e., to finding outputs. This avoids solving a system of approximate differential equations. A further simplification of these equations leads finally to a system of algebraic equations for determining outputs.

Taking account of what has been said, we replace the initial mathematical model by the following heuristic model:

$$(1+A)\frac{dv}{dx}-f(v,\ v')=0,\quad (1+A')\frac{dv'}{dx}-f'(v,\ v')=0, \tag{16}$$

where v and v' are approximate potential functions, and the boundary conditions for Eqs. (16) are given in the form

$$(1+A)\,v(0)-Av(1)=u_b\quad (1+A')\,v'(0)-A'v'(1)=u_b'. \tag{17}$$

Here the parameters are those of the exact model with the same meaning. It is clear that these equations and boundary conditions satisfy the requirements formulated and that the integration of the problem is very much simpler and can generally be performed in closed form. It should be noted that compatibility of the boundary conditions is assured for functions f and f' which are of practical interest.

We now use Eqs. (16) and (17) to find approximate values of the potentials at the system exit. In all the source functions in (16) we replace a potential whose derivative does not enter the equation by its average value, defined in the following way:

$$\bar{v}=\int_0^1 v(x)\,dx,\quad \bar{v'}=\int_0^1 v'(x)\,dx. \tag{18}$$

The equations are integrated by making appropriate substitutions. We treat the average values substituted into these equations as constants. After integrating and using (17) we obtain the following simple relations:

$$(1+A)\int_{\frac{u_b+Av(1)}{1+A}}^{v(1)}\frac{dv}{f(v,\ \bar{v'})}=1,\quad (1+A')\int_{\frac{u_b'+A'v'(1)}{1+A'}}^{v'(1)}\frac{dv'}{f'(\bar{v},\ v')}=1. \tag{19}$$

Two more relations are needed to determine the average values since Eqs. (18) do not suffice. After transformations and integration we obtain

$$(1+A)\int_{\frac{u_b+Av(1)}{1+A}}^{v(1)}\frac{v\,dv}{f(v,\ \bar{v'})}=\bar{v},\quad (1+A')\int_{\frac{u_b'+A'v'(1)}{1+A'}}^{v'(1)}\frac{v'dv'}{f'(\bar{v},\ v')}=\bar{v'}. \tag{20}$$

Equations (19) and (20) form a system of algebraic equations from which the potentials at the system exit can be found after eliminating the average values.

It should be noted, however, that the values of the potentials at the exit found in this way do not agree with the values obtained by solving the mathematical problems (16) and (17) and substituting x = 1. This results from the fact that in integrating Eqs. (16) the separate potentials were sometimes treated as variables and sometimes as constants. Therefore, although the chosen constants are average values, we have not succeeded in finding the correct values of the potentials at the exit. Consequently, the approximation involves replacing the basic original model by an approximate model, and then obtaining an approximate solution of the approximate model. This fact is taken into account later in estimating the error of the approximation.

To estimate the error of the approximation method it is first necessary to reduce (16) and (17) to the form of Eqs. (15). Equations (15) are obtained from (2), which describe the original model, and from the boundary conditions for the original model. A rigorous analysis of the differences in the conclusions which follow from the solutions of the exact and approximate problems is hardly justified. Therefore, on the basis of the method described above we seek a solution of Eqs. (16) in the form

$$v(x)=1\cdot E(x),\quad v'(x)=1\cdot E'(x). \tag{21}$$

Substituting this into (16) and integrating, we obtain

$$E(x)=\int_0^x \frac{f(v(\xi),\ v'(\xi))}{1+A}\,d\xi+E_0,$$
$$E^1(x)=\int_0^x \frac{f'(v(\xi),\ v'(\xi))}{1+A'}\,d\xi+E_0', \tag{22}$$

where E_0 and E_0' are constants. Now by substituting (22) into (17) and using (21) we obtain expressions for the constants:

$$E_0=u_b+\frac{A}{1+A}\int_0^1 f(v(\xi),\ v'(\xi))\,d\xi,$$
$$E_0'=u_b'+\frac{A'}{1+A'}\int_0^1 f'(v(\xi),\ v'(\xi))\,d\xi. \tag{23}$$

Finally, substituting (23) back into (22) and (21) and transforming the result, we obtain the following system of integral equations:

$$v(x)=u_b+\int_0^x f(v(\xi),\ v'(\xi))\,d\xi+\frac{A}{1+A}\int_x^1 f(v(\xi),\ v'(\xi))\,d\xi,$$
$$v'(x)=u_b'+\int_0^x f'(v(\xi),\ v'(\xi))\,d\xi+\frac{A'}{1+A'}\int_x^1 f'(v(\xi),\ v'(\xi))\,d\xi. \tag{24}$$

Equations (24) are equivalent to problem (16), (17) in the same way as Eqs. (15) are equivalent to problem (2), (8), (10).

We try to characterize the error of the approximation method used within the interval of interest by upper estimates of the following quantities:

$$\delta(x)=|u(x)-v(x)|,\quad \delta'(x)=|u'(x)-v'(x)|. \tag{25}$$

By comparing the corresponding equations of systems (15) and (24) we obtain the following equations for the quantities in (25):

$$\delta(x)=\Big|\int_0^x [f(u(\xi),\ u'(\xi))-f(v(\xi),\ v'(\xi))]\,d\xi+\int_x^1 [e^{\frac{x-\xi}{A}}f(u(\xi),\ u'(\xi))-\frac{A}{1+A}\,f(v(\xi),\ v'(\xi))]\,d\xi\Big|,$$
$$\delta'(x)=\Big|\int_0^x [f'(u(\xi),\ u'(\xi))-f'(v(\xi),\ v'(\xi))]\,d\xi+\int_x^1 [e^{\frac{x-\xi}{A'}}f'(u(\xi),\ u'(\xi))-$$
$$-\frac{A'}{1+A'}\,f'(v(\xi),\ v'(\xi))]\,d\xi\Big|. \tag{26}$$

Since we seek an upper limit we can obtain from Eqs. (26) the inequalities

$$\delta(x)\leqslant\Big|\int_0^x [f(u(\xi),\ u'(\xi))-f(v(\xi),\ v'(\xi))]\,d\xi\Big|+\Big|\int_x^1 [e^{\frac{x-\xi}{A}}f(u(\xi),\ u'(\xi))-$$
$$-\frac{A}{1+A}\,f(v(\xi),\ v'(\xi))]\,d\xi\Big|\leqslant\int_0^x |f(u(\xi),\ u'(\xi))-f(v(\xi),\ v'(\xi))|\,d\xi+$$
$$+\int_x^1 |e^{\frac{x-\xi}{A}}f(u(\xi),\ u'(\xi))-\frac{A}{1+A}\,f(v(\xi),\ v'(\xi))|\,d\xi, \tag{27}$$

$$\delta'(x) \leqslant \left| \int_0^x [f'(u(\xi), u'(\xi) - f'(v(\xi), v'(\xi))] d\xi \right| + \left| \int_x^1 [e^{\frac{x-\xi}{A'}} f'(u(\xi), u'(\xi)) - \right.$$

$$\left. - \frac{A'}{1+A'} f'(v(\xi), v'(\xi))] d\xi \right| \leqslant \int_0^x |f'(u(\xi), u'(\xi)) + f'(v(\xi), v'(\xi))| d\xi +$$

$$+ \int_x^1 [e^{\frac{x-\xi}{A'}} f'(u(\xi), u'(\xi)) - \frac{A'}{1+A'} (v(\xi), v'(\xi))| d\xi.$$

Further transformation of Eqs. (27) requires the following additional assumptions:

a) The source functions f and f' satisfy the first-order Lipshits condition with respect to both their arguments, and the constants appearing in the specific expressions for these functions can be specified from independent physical or mathematical considerations.

b) The maximum absolute values of the source functions for the process can be indicated from physical or mathematical considerations.

Thus, on the basis of what has been said we use the following relations for a further estimate of the error of the approximation:

$$|f(u, u') - f(v, v')| \leqslant K(|u-v| + |u'-v'|),$$
$$|f'(u, u') - f'(v, v')| \leqslant K'(|u-v| + |u'-v'|), \quad (28)$$

where K and K' are constants. If, for example, the potentials which occur can be written in simplex form, for which $0 \leq u, u'; v, v' \leq 1$, these constants satisfy the following estimates:

$$K = \max_{0 \leqslant u, u' \leqslant 1} \left[\left| \frac{\partial f}{\partial u} \right|, \left| \frac{\partial f}{\partial u'} \right| \right],$$
$$K' = \max_{0 \leqslant u, u' \leqslant 1} \left[\left| \frac{\partial f'}{\partial u} \right|, \left| \frac{\partial f'}{\partial u'} \right| \right]. \quad (29)$$

Only crude estimates can be made of the second terms of the error

$$|e^{\frac{x-\xi}{A}} f(u, u') - \frac{A}{1+A} f(v, v')| \leqslant 2M,$$
$$|e^{\frac{x-\xi}{A'}} f(u, u') - \frac{A'}{1+A'} f'(v, v')| \leqslant 2M', \quad (30)$$

where the constants M and M' can be adequately estimated from similar arguments:

$$M = \max_{0 \leqslant u, u' \leqslant 1} [|f(u, u')|], \quad M' = \max_{0 \leqslant u, u' \leqslant 1} [|f'(u, u')|]. \quad (31)$$

Now, comparing (27), (25), (28), and (30), and ordering the results, we obtain the following system of integral inequalities for estimating the error:

$$\delta \leqslant K \int_0^x (\delta + \delta') + 2M(1-x), \quad \delta' \leqslant K' \int_0^x (\delta + \delta') + 2M'(1-x). \quad (32)$$

We introduce the following integral errors which are equivalent to (25) with respect to the characteristics of the accuracy of the method:

$$\Delta = \int_0^x \delta, \quad \Delta' = \int_0^x \delta'. \quad (33)$$

Then, by comparing (32) and (33) we can transform the system of integral inequalities into a system of differential inequalities

$$\frac{d\Delta}{dx} \leqslant K\Delta + K\Delta' + 2M(1-x), \quad \frac{d\Delta'}{dx} \leqslant K'\Delta + K'\Delta' + 2M'(1-x). \quad (34)$$

In addition, the following is known with respect to (33):

$$\Delta(x=0) = 0, \quad \Delta'(x=0) = 0. \quad (35)$$

Since the error of the approximation method can be completely characterized by the sum of the errors in determining the potentials, and since such a sum is an upper estimate, from now on we consider the summation of the integral errors. We add the quantities in (34):

$$\frac{d}{dx}(\Delta+\Delta') \leqslant (K+K')(\Delta+\Delta') + 2(M+M')(1-x). \tag{36}$$

We regroup (36) and multiply both sides of the inequality obtained by a positive definite function ρ which is so far unknown:

$$\rho\frac{d}{dx}(\Delta+\Delta') - (K+K')\rho(\Delta+\Delta') \leqslant 2(M+M')\rho(1-x). \tag{37}$$

We now try to choose ρ so that the left-hand side of (37) contains the derivative of the product of two functions. It is easy to see that this requirement is satisfied if ρ satisfies the equation $d\rho/dx = -(K + K')\rho$; i.e., $\rho = e^{-(K+K')x}$. Substituting the result of this operation into (37) we obtain

$$\frac{d}{dx}(\Delta+\Delta')\,e^{-(K+K')x} \leqslant 2(M+M')(1-x)\,e^{-(K+K')x}. \tag{38}$$

Since the integration of both sides of any inequality over definite limits preserves the inequality, Eq. (38) can be integrated from 0 to x:

$$[(\Delta+\Delta')\,e^{-(K+K')\xi}]_2^x \leqslant \left[2\frac{M+M'}{K+K'}\left(\xi - \frac{K+K'-1}{K+K'}\right)e^{-(K+K')\xi}\right]_0^x. \tag{39}$$

Substituting the limits of integration into (39), using (35), and multiplying both sides of the result by the positive definite function $e^{(K+K')x}$, we obtain

$$\Delta+\Delta' \leqslant 2\frac{M+M'}{K+K'}\left[x + \frac{K+K'-1}{K+K'}\left(e^{(K+K')x}-1\right)\right]. \tag{40}$$

Finally, we substitute $x = 1$ in Eq. (40). This gives an upper estimate for the approximation method, and this means an upper limit also for the integral error of the values of the potentials at the system exit calculated from Eqs. (19) and (20).

The difference between the established error and the actual error will be smaller the more accurately the estimate in (30) is performed. However, the main difficulties in improving this estimate are first the fact that a further solution of the approximate mathematical model gives only the values of the potentials at the exit, but does not lead to any information for the range $0 \leq x < 1$ under investigation. Secondly, in the integration of the equations the effect of periodic fixation of separate variables can be established only for known functions f and f'. Therefore the functions f and f' together with their absolute limiting values must be estimated. At the same time the estimate unfortunately becomes relatively crude and, for example, the dependence of the error on the degree of mixing vanishes.

A complex approximation will be very much more accurate. For example, a direct comparison of (15) and (24) shows that the solutions agree, and this means that the values of the potentials at the exit also agree in the limit $A \to \infty$; i.e., the error will be zero. The approximate mathematical model will be exact in the limit $A \to 0$ also. In any case it can be established from Eq. (40) that the error cannot diverge for any functions f and f' which are bounded or monotonic in the range in question.

NOTATION

A, A', A_1, A_1', A_2, A_2', dimensionless mixing coefficients of corresponding phases or in corresponding intervals; C_{11}, C_{11}', C_{12}, C_{12}', C_{21}, C_{21}', C_{22}, C_{22}', integration constants; E, E', auxiliary functions; E_0, E_0', integration constants; f^*, f'^*, f, f', source functions of separate phases and their transformed forms; F, F', transfer cross sections for separate phases; $\mathcal{K}$, $\mathcal{K}'$, constant mixing coefficients of separate phases; L, linear dimension of system; M, M', constants; u, u', u_1, u_1', u_2, u_2', u_b, u_b', potentials in separate phases or intervals or potentials far from system entrance; V, V', bulk flow velocities of separate phases; v, v', $\bar{v}$, $\bar{v}'$, approximate values of potentials in separate phases and their average values; x^*, x, length coordinate and its transformed value; Γ_1, Γ_1', Γ_2, Γ_2', auxiliary functions; Γ_{10}, Γ_{10}', Γ_{20}, Γ_{20}', integration constants; δ, δ', error functions; Δ, Δ', integral errors; ξ, integration parameter; ρ, auxiliary function.

LITERATURE CITED

1. T. Blikle and T. Vaida, Paper on Technical Days of Chemistry 1976, Keszthely (1976).
2. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, Transport Phenomena, Wiley, New York (1960).

3. J. F. Wehner and R. H. Wilhelm, Chem. Eng. Sci., 6, 89 (1956).
4. A. Kneschke, Differentialgleichungen, Vol. 1, Teubner Verlagsgesellschaft, Leipzig (1965).

THE DYNAMIC PRESSURE FROM A COMPRESSED ARC IN METAL CUTTING

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Direct spectroscopic techniques applied to plasma metal cutting show that the dynamic pressure is dependent on the working conditions in the plasma source.

The metal melts during plasma cutting mainly in the zone of the periodically moving anode spot; the molten metal is transported by the hot gas, and it is therefore important to examine the dynamic pressure exerted by the flow in relation to source conditions in order to choose the optimum cutting conditions [1]. The dynamic characteristics have been examined [2] by means of contact transducers in a system where the metal was a water-cooled copper disk. More reliable determinations have been provided by optical spectroscopy, since these provide much higher time resolution.

We used the system of [1], which contained a power supply and an OPR-11 control unit, as well as a PMR-6 plasma source, where the workpiece was a strip of Kh18N10T steel of thickness 10-50 mm. The range of working conditions was as follows: current I = 150-300 A, voltage U = 165-200 V, gas flow rate (nitrogen) R_g = 50-125 liters/min, diameter of cathode nozzle d_c = 2.5-3.5 mm, and length of open zone of plasma h = 6 mm.

The data of [1] were used to determine the dynamic head ρV^2 (where ρ is density and V is velocity) from the velocity V averaged over the cross section and spectroscopic measurement of the temperature, which is required in order to calculate ρ. The spectra were recorded photographically with an ISP-30. The region 300-565 nm from the open plasma consists of a continuum and the lines from nitrogen atoms and ions (Fig. 1). Spectral lines due to the electrodes (cathode and anode) are absent. This means that the open zone can be treated as free from impurities, so spectroscopic techniques can be used to determine the temperature from the nitrogen line strengths on the assumption that the plasma consists of nitrogen only. The transparent-plasma approximation was used with the NII 359.3 and 504.5 nm lines, which show no reabsorption under these conditions [3, 4]. The correction was applied for the plasma nonuniformity by solving Abel's integral equation by the method of [5] (NII 359.3; 360.9; 391.9; 517.5 nm); the plasma composition and the transition probabilities were taken from [6, 7, 9]. The relative error in determining the temperature from the absolute intensities did not exceed 5%.

Figure 2 shows the radial temperature distributions at 2 mm from the end of the nozzle for a nitrogen flow rate of 50 liters/min for discharge currents of 150, 200, 250, and 300 A; the plasma had a higher temperature gradient. The maximum temperature attained under these conditions was 19,000°K for I = 200 A; then the temperature at 1.5 mm from the axis was 14,000°K. It was difficult to measure the temperature at more remote points on account of the poor dynamic range in the photographic recording. In addition, the central ray was used to determine the temperature from the relative intensities of the NII 359.3, 360.9, 391.9, and 517.5 nm lines. The discrepancy between the axial temperature and the temperature from the central ray was less than 3000°K.

These results do not agree with those of [8], where a rise in axial temperature with arc current was reported. The temperature attained 30,000°K for I = 300 A. Possible reasons for the discrepancy are as follows. First, the rise in axial temperature with the current was slight and lay within the error of experiment, and the workpiece was replaced [8] by a water-cooled anode, which could affect the plasma parameters. Also, the system had a poor

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